THREE-DIMENSIONAL TWISTED SUPERSYMMETRY ON THE LATTICE

Dario van den Berg, Anosh Joseph
2423951@students.wits.ac.za

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1. WHY STUDY 3D $\mathcal{N} = 4$ SYM ON THE LATTICE?

- Supersymmetry (SUSY) is one of the most compeling extensions of the Standard Model and appears in many theories beyond it.
- Strongly coupled quantum field theories are difficult to solve analytically.
- Lattice gauge theory discretizes spacetime, enabling non-perturbative numerical simulations.

The Lattice Idea

Continuous spacetime → Discretize spacetime → Numerical simulation

2. FROM 6D $\mathcal{N}$ = 1 to 3D $\mathcal{N}$ = 4 VIA TOPOLOGICAL TWISTING

Start with 6D Euclidean $\mathcal{N}$ = 1 Super Yang-Mills theory

6D $\mathcal{N}$ = 1 SYM
$\Downarrow$
3D $\mathcal{N}$ = 4 SYM

Three dimensions are compactidied (or ignored), producing three scalars and reorganizing supersymmetry.

R-symmetry decomposition

$SO(6)_E \rightarrow SU(2)_E \times SU(2)_N$

The full symmetry before twist:

$SU(2)_E \times SU(2)_R \times SU(2)_N$

Blau-Thompson Twist

$SU(2)' = \mathrm{diag}\,(SU(2)_E \times SU(2)_N)$

Twisting is a change of variables that combines spacetime rotations with an internal symmetry.

Twisted fields (geometric objects)

A_M	$\rightarrow A_m$ (gauge field 1-form)
(scalars from A_M)	$\rightarrow \eta$ (0-form)
(fermions)	$\rightarrow \Psi_m$ (1-form)
(fermions)	$\rightarrow \chi_{mn}$ (2-form)
(fermions)	$\rightarrow \theta_{mnp}$ (3-form)

Supercharges reorganize as

$\{Q, Q_m, Q_{mn}, Q_{mnp}\}$ with $Q^2 = 0$

3.TWISTED SUSY TRANSFORMATIONS

$$\delta_k = \delta_{ko}Q + \delta_{ka}Q_a + \delta_{kab}Q_{ab} + \delta_{abc}Q_{abc}$$

$$\delta_k A_m = \delta_{ko}\Psi_m + \frac{1}{2}\delta_{km}\eta + \delta_{kab}(\delta_{am}\Psi_b - \delta_{bm}\Psi_a)$$

$$\delta_k \bar{A}_m = -\delta_{ka}\chi_{am} + \delta_{kabc}\epsilon_{abc}\Psi_m$$

$$\delta_k \Psi_m = \delta_{ka}(\frac{1}{2}\delta_{am}d_a + (1-\delta_{am})[D_a, D_m])$$

$$\delta_k \chi_{mn} = -\delta_{ko}F_{mn} + \frac{1}{2}\delta_{kab}(\delta_{am}\delta_{bn} - \delta_{an}\delta_{bn})d_{ab} + \delta_{kabc}\epsilon_{abcd}d_m$$

$$\delta_k \eta = \delta_{ko}d + 2\delta_{kab}F_{ab} + \delta_{kabc}\theta_{abc}$$

$$\delta_k B_{abc} = -\delta_{ko}\theta_{abc} - \delta_{kd}\chi_{bc}$$

$$\delta_k \theta_{abc} = -\delta_{kd}F_{bc}$$

Indices $a, b, c, \ldots$ = 1, 2, 3 label the three twisted spatial directions.

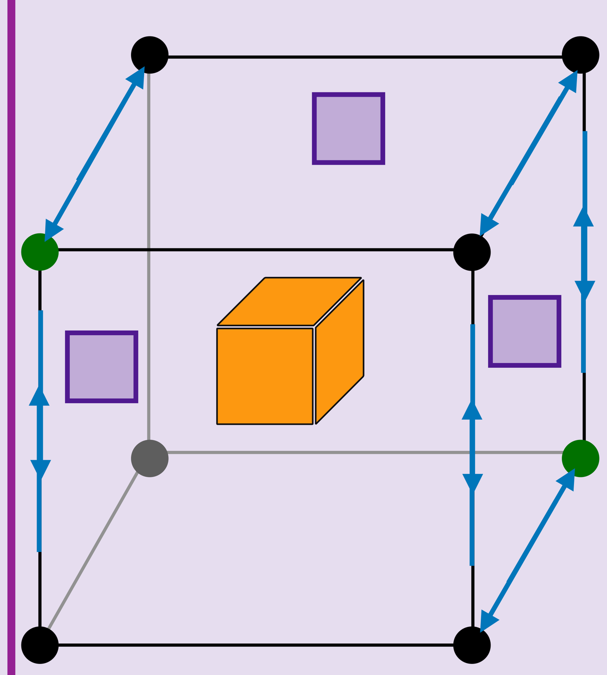
4. PLACING FIELDS ON THE CUBIC LATTICE

Sites(0-form) η

Links(1-form) $A_m, \bar{A}_m, \Psi_m$

Plaquettes(2-form) χ_{mn}

Cubes (3-form) B_{abc}, θ_{abc}



$\eta(n) \rightarrow G(n)\eta(n)G^\dagger(n)$

$A_m(n) \rightarrow G(n)A_m(n)G^\dagger(n + \bar{\mu}_m)$

$\bar{A}_m(n) \rightarrow G(n + \bar{\mu}_m)\bar{A}_m(n)G^\dagger(n)$

$\Psi_m(n) \rightarrow G(n)\Psi_m(n) + G^\dagger(n + \bar{\mu}_m)$

$\chi_{mn}(n) \rightarrow G(n + \bar{\mu}_m + \bar{\mu}_n)$ $\chi_{mn}(n)G^\dagger(n)$

$B_{abc}(n) \rightarrow G(n + \bar{\mu}_a + \bar{\mu}_b + \bar{\mu}_c)$ $B_{abc}(n)G^\dagger(n)$

$\theta_{abc}(n) \rightarrow G(n + \bar{\mu}_a + \bar{\mu}_b + \bar{\mu}_c)+$ $\theta abc(n)G^\dagger(n)$

Fields live on the lattice elements that match their geometric character. This assignment,together with the gauge transformations,preserve local $U(\mathcal{N})$ gauge invariance at ∞ spacing.

5.EXACT SCALAR SUPERCHARGE Q

$QA_m(n) = \Psi_m(n)$
$Q\bar{A}_m(n) = 0$
$Q\Psi_m(n) = 0$
$Q\chi_{mn}(n) = -F^\dagger_{mn} = -F_{mn}(n)$
$Q\eta(n) = d(n)$
$Qd(n) = 0$
$QB_{abc}(n) = -\theta_{abc}(n)$
$Q\theta_{abc}(n) = 0$

✓ The scalar supercharge Q maps each field to another field on the same lattice element.

✓ It commutes with local gauge transformations exactly.

$Q^2 = 0$
(exact at any lattics spacing)

6. WHY ONLY Q SURVIVES ON THE LATTICE

Non-scalar supercharges map between incompatible lattice objects. Examples (3-form charge):

$$Q_{abc}\bar{A}_d(n + \hat{\mu}_d, n), n) = \epsilon_{abcd}\Psi_d(n, n + \hat{\mu}_d)(n, n + \hat{\mu})$$

$$n + \hat{\mu}_d \longleftarrow n \quad \neq \quad n \longrightarrow n + \hat{\mu}_d$$

✗ Differential lattice endpoints ⇒ different gauge transformation properties.

✗ Non-scalar transformations fail to commute with gauge invariants at non-zero lattice spacing (Elizur's theorem ⇒ trivial Ward identities).

Therefore, only the scalar charge Q remains an exact symmetry.

Q	✓ survives
Q_m	✗ broken
Q_{mn}	✗ broken
Q_{mnp}	✗ broken

Main Message: Topological twisting reorganizes supersymmetry into geometric objects that fit naturally on the lattice. This allows the nilpotent scalar supercharge Q to remain exact at finite lattice spacing, guaranteeing full 3D $\mathcal{N}$=4 theory is recovered in the continuum limit.

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[4] J. Kato and T. S Wu, *Phys. Lett B* **665**, 12 (2008).