Distinguishing $\frac{1}{2}$-BPS states using symmetric polynomials

Garreth Kemp

Department of Physics, University of Johannesburg

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AdS/CFT in one slide

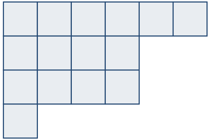
Quantum gravity on $AdS_5 \times S^5$

$$\Updownarrow$$

$\mathcal{N} = 4$ super Yang–Mills theory with $U(N)$ gauge group

- AdS/CFT translates questions about quantum gravity into questions in a gauge theory.
- In favourable sectors, the gauge theory problem becomes exactly computable.
- Using representation theory to ask how precisely a state can be identified/distinguished.

Half-BPS states and Young diagrams



- In the $\frac{1}{2}$-BPS sector of $\mathcal{N} = 4$ SYM, states are labelled by Young diagrams R.
- These diagrams label irreducible representations of $U(N)$ and of symmetric groups.
- R encode the eigenvalues of N conserved charges.
- Through AdS/CFT, this data is related to dual gravitational configurations.

BPS state $\longleftrightarrow$ Young diagram $\longleftrightarrow$ N conserved charges $\longleftrightarrow$ geometry

The guiding question

How much information is needed to distinguish two half-BPS states?

Physics side

- Conserved charges in the gauge theory
- Multipole moments in the dual geometry
- Information needed to identify a state

Mathematics side

- Young diagrams
- Symmetric group representations
- Symmetric polynomials

This talk is about a clearer mathematical formulation for this problem.

The conserved charges as character data

The relevant conserved charges can be expressed using central elements of the symmetric group algebra.

Cycle operators

$$T_k = \sum_{\sigma \in S_n \,:\, \sigma \text{ has one } k\text{-cycle}} \sigma.$$

For a Young diagram R, the data

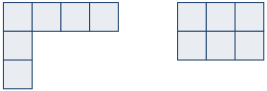
$$\mathcal{X}_k^{(R)} = \{\widehat{\chi}_R(T_2), \widehat{\chi}_R(T_3), \ldots, \widehat{\chi}_R(T_k)\}$$

can be used to distinguish R from other Young diagrams.

Question: how large must k be before all states are distinguished?

Previous work: a generalized dominance ordering

Previous work studied when two Young diagrams are not distinguished by the first few charges. [Kemp arXiv:2305.06768]



- Young diagrams of n boxes are partially ordered by dominance, or majorization.
- Some pairs are incomparable.

Key idea from [Kemp arXiv:2305.06768]

The amount of incomparability between two diagrams controls how many charges may fail to distinguish them.

Dominance and incomparability

For two partitions $X = (x_1, x_2, \ldots)$ and $Y = (y_1, y_2, \ldots)$, dominance compares cumulative row sums:

$$X \prec Y \iff \sum_{i=1}^{K} x_i \leq \sum_{i=1}^{K} y_i \quad \text{for all } K.$$

Comparable

The inequality signs all point the same way.

$$< \quad < \quad < \quad <$$

Incomparable

The inequality signs change direction.

$$< \quad > \quad > \quad <$$

The number of direction changes is denoted k_{swap}.

The conjecture

Suppose two diagrams Y_1, Y_2 have identical charge data up to $T_{k_{\mathrm{deg}}}$,

$$\mathcal{X}_{k_{\mathrm{deg}}}^{(Y_1)} = \mathcal{X}_{k_{\mathrm{deg}}}^{(Y_2)},$$

but are distinguished by the next charge.

Conjecture

$$k_{\mathrm{deg}}^{(Y_1,Y_2)} \leq k_{\mathrm{swap}}^{(Y_1,Y_2)} + 1.$$

In words:

more incomparability allows more degeneracy,
but only up to a controlled bound.

Why this is useful

The conjecture gives a practical upper bound on the number of charges required to distinguish two states.

Consequence

If

$$k_*^{(Y_1,Y_2)} = \text{minimum number of cycle operators needed},$$

then

$$k_*^{(Y_1,Y_2)} \leq k_{\mathrm{swap}}^{(Y_1,Y_2)} + 2.$$

Converts a difficult character computation into a much simpler combinatorial estimate from Young diagrams.

$\boxed{\text{How to prove the conjecture?}}$

The old formulation: power sums

In [Kemp] the character degeneracy problem was reformulated using power sum symmetric polynomials.

Power sums evaluated on a state:

$$p_q(\widetilde{Y}) = \sum_i \widetilde{y}_i^{\,q} \qquad \mathcal{P}_k^{(Y)} = \{p_1(\widetilde{Y}), p_2(\widetilde{Y}), \cdots, p_k(\widetilde{Y})\}$$

The following two statements are equivalent:

$$\mathcal{X}_k^{(Y_1)} = \mathcal{X}_k^{(Y_2)} \qquad \Longleftrightarrow \qquad \mathcal{P}_k^{(Y_1)} = \mathcal{P}_k^{(Y_2)}$$

$$\boxed{\begin{array}{c}\text{character}\\ \text{degeneracy}\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{power sum}\\ \text{degeneracy}\end{array}}$$

Problem: the identities become cumbersome at higher degree.

The new formulation: elementary symmetric polynomials

This work rewrites the same data using elementary symmetric polynomials.

$$e_q(x_1, \dots, x_d) = \sum_{1 \le a_1 < a_2 < \cdots < a_q \le d} x_{a_1} x_{a_2} \cdots x_{a_q}.$$

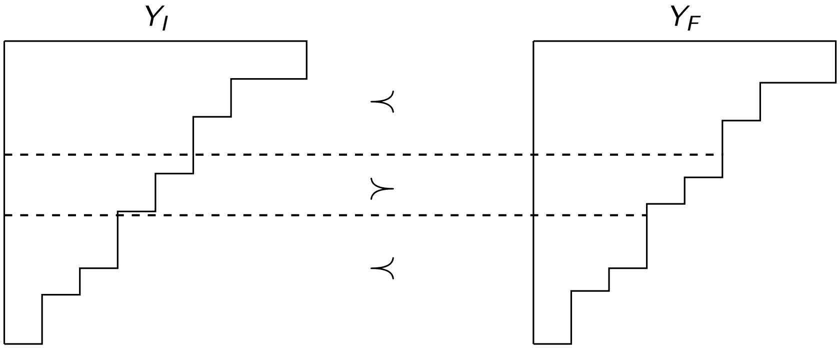
characters $\longleftrightarrow$ power sums p_q $\longleftrightarrow$ elementary polynomials e_q

The bridge is provided by the Newton-Girard identities.

Case study: $k_{\text{swap}}^{(Y_I, Y_F)} = 2$

For $k_{\text{swap}}^{(Y_I, Y_F)} = 2$. Imagine 3 regions:

$$Y_I^{(1)} \prec Y_F^{(1)}. \qquad Y_I^{(2)} \succ Y_F^{(2)}, \qquad Y_I^{(3)} \prec Y_F^{(3)}$$



Case study: $k_{\text{swap}}^{(Y_I,Y_F)} = 2$

Imagine performing a series of $1 - \square$ moves to go from Y_I to Y_F:

$$Y_I \to Y_1 \to Y_2 \to Y_3 \to \cdots \to Y_F.$$

Conjecture says

$$k_{deg} \leq 3.$$

Assume the constraints: $e_2(\widetilde{Y}_I) = e_2(\widetilde{Y}_F)$, and $e_3(\widetilde{Y}_I) = e_3(\widetilde{Y}_F)$,

$$\Delta e_2^{(Y_I,Y_F)} = 0, \qquad \Delta e_3^{(Y_I,Y_F)} = 0.$$

Prove:

$$e_4(\widetilde{Y}_I) \neq e_4(\widetilde{Y}_F), \qquad \Delta e_4^{(Y_I,Y_F)} \neq 0.$$

Why the e_q's help

The elementary symmetric polynomials seem better suited to this problem.

$$\Delta e_q^{(Y_I, Y_F)} = \sum_{i=0}^{N} \Delta e_q^{(Y_i, Y_{i+1})}$$

Key identity

For a $1 - \square$ move from row k to l; $Y_{\mathrm{red}} = Y \setminus \{y_k, y_l\}$

$$\Delta e_q = \Delta e_2 \times e_{q-2}(Y_{\mathrm{red}}).$$

Analytic progress in this work

Simple case: Moving one $\square$ in each region.

$$
\begin{aligned}
\Delta e_2^{(Y_I,Y_F)} &= \Delta e_2(1) + \Delta e_2(2) + \Delta e_2(3), \\
\Delta e_3^{(Y_I,Y_F)} &= e_1(1)\Delta e_2(1) + e_1(2)\Delta e_2(2) + e_1(3)\Delta e_2(3), \\
\Delta e_4^{(Y_I,Y_F)} &= e_2(1)\Delta e_2(1) + e_2(1)\Delta e_2(2) + e_2(1)\Delta e_2(3).
\end{aligned}
$$

Strategy: Assume $\Delta e_2^{(Y_I,Y_F)} = \Delta e_3^{(Y_I,Y_F)} = \Delta e_4^{(Y_I,Y_F)} = 0$:

$$
\begin{pmatrix} 1 & 1 & 1 \\ e_1(1) & e_1(2) & e_1(3) \\ e_2(1) & e_2(2) & e_2(3) \end{pmatrix}
\begin{pmatrix} \Delta e_2(1) \\ \Delta e_2(2) \\ \Delta e_2(3) \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

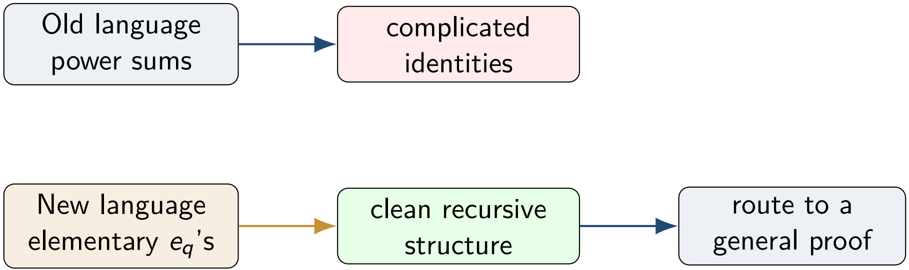
Analytic progress in this work

$$\mathrm{Det}\begin{pmatrix} 1 & 1 & 1 \\ e_1(1) & e_1(2) & e_1(3) \\ e_2(1) & e_2(2) & e_2(3) \end{pmatrix} \neq 0.$$

This approach generalises:

$$k_{\text{swap}}^{(Y_I, Y_F)} = 3, \qquad \text{Study } \ \mathrm{Det}\begin{pmatrix} 1 & 1 & 1 & 1 \\ e_1(1) & e_1(2) & e_1(3) & e_1(4) \\ e_2(1) & e_2(2) & e_2(3) & e_2(4) \\ e_3(1) & e_3(2) & e_3(3) & e_3(4) \end{pmatrix}.$$

What is the main contribution?



Main message

This work identifies a more natural algebraic formulation that may scale to the general conjecture.

Outlook

The next goals are:

- Extend the elementary-symmetric-polynomial method to arbitrary k_{swap}.
- Prove the conjectured bound
$$k_{\mathrm{deg}} \leq k_{\mathrm{swap}} + 1.$$
- Understand the implications for the centre $Z(\mathbb{C}[S_n])$.
- Translate the result back into the physics question of distinguishing half-BPS states.

A combinatorial bound on Young diagrams may encode a bound on distinguishability in quantum gravity.

Summary

1. Half-BPS states are labelled by Young diagrams.
2. Conserved charges can be encoded as characters of central elements of $\mathbb{C}[S_n]$.
3. The previous work proposed a bound controlled by the incomparability measure k_{swap}.
4. This paper reformulates the problem using elementary symmetric polynomials.
5. The new formulation makes the analytic arguments simpler and points toward a general proof.

Thank you

Analytic progress in this work

We use the elementary symmetric polynomials to rederive the earlier analytic results.

$k_{\mathrm{swap}} = 0$

Comparable diagrams are already distinguished by T_2.

$k_{\mathrm{swap}} = 1$

If e_2 is degenerate, then e_3 separates the diagrams.

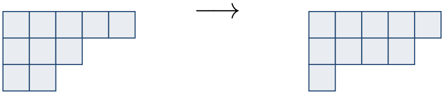
Limited $k_{\mathrm{swap}} = 2$

If e_2 and e_3 are degenerate in a minimal case, then e_4 separates the diagrams.

The proofs are shorter and may be extendable to higher k_{swap}!

Backup: one-box moves

A one-box move shifts one box from the end of one row to another row, while preserving the Young diagram conditions.



The elementary symmetric polynomials respond to such moves through the clean relation

$$\Delta e_q = \Delta e_2 \, e_{q-2}(\widetilde{Y}_{\mathrm{red}}),$$

where $\widetilde{Y}_{\mathrm{red}}$ is obtained by removing the two rows involved in the move.