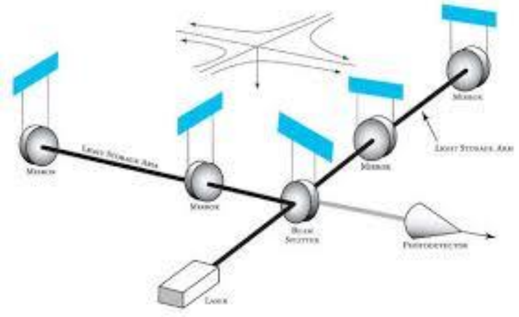
Evaluation of AIM using Saad and Batic approaches

UNIVERSITY OF JOHANNESBURG

https://www.uj.ac.za/brand/brand-fundamentals/

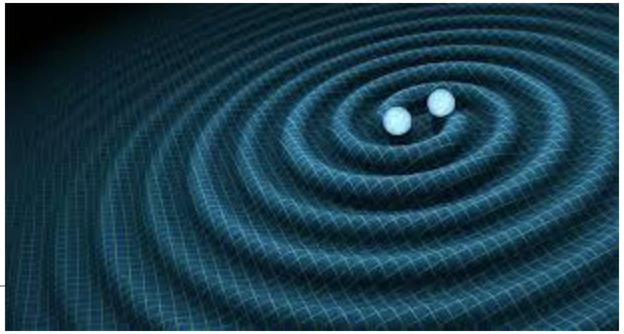


MSC RESEARCH - JULY 2026

TAL LESHEM

SUPERVISOR: PROF ALAN CORNELL

Regge Wheeler equation



$$ds^2 = (1 - 2m/r)dt^2 - (1 - 2m/r)^{-1}dr^2 - r^2(d\theta^2 + sin^2 d\phi^2)$$

$$\frac{d^2\psi(r)}{dx^2} + \left[\omega^2 - V_{s,l}(r)\right]\psi(r) = 0$$

$$V_{s,l}(r) = f(r)\left[\frac{l(l+1)}{r^2} + (1 - s^2)\left(\frac{2m}{r^3} - \frac{(4 - s^2)\Lambda}{6}\right)\right]$$

T. Regge and J. A. Wheeler, "Stability of a Schwarzschild singularity," *Phys. Rev.*, vol. 108, pp. 1063–1069, 1957.

Asymptotic Iteration Method (AIM)

H. T. Cho, A. S. Cornell, J. Doukas, T. R. Huang, and W. Naylor, "A New Approach to Black Hole Quasinormal Modes: A Review of the Asymptotic Iteration Method," *Adv. Math. Phys.*, vol. 2012, p. 281705, 2012.

$$y'' = \lambda_0(x)y' + s_0(x)y$$

$$y^{(n+2)} = \lambda_n(x)y' + s_n(x)y.$$

$$\lambda_n = \lambda'_{n-1} + s_{n-1} + \lambda_0\lambda_{n-1},$$

$$s_n = s'_{n-1} + s_0\lambda_{n-1}.$$

Asymptotic assumption:

$$\frac{s_n}{\lambda_n} = \frac{s_{n-1}}{\lambda_{n-1}} = \alpha(x)$$

N. Saad, R. L. Hall, and H. Ciftci, "Asymptotic iteration method for eigenvalue problems," *J. Phys. A*, vol. 36, no. 47, p. 11807, 2003.

ℓ	n	ω_{Leaver}	ω_{AIM} (after 15 iterations)	ω_{WKB}
2	0	0.3737–0.0896i [*]	0.3737–0.0896i (<0.01%)(<0.01%)	0.3732–0.0892i (–0.13%)(0.44%)[*]
	1	0.3467–0.2739i	0.3467–0.2739i (<0.01%)(<0.01%)	0.3460–0.2749i (–0.20%)(–0.36%)
	2	0.3011–0.4783i	0.3012–0.4785i (0.03%)(–0.04%)	0.3029–0.4711i (0.60%)(1.5%)
	3	0.2515–0.7051i	0.2523–0.7023i (0.32%)(0.40%)	0.2475–0.6703i (–1.6%)(4.6%)
3	0	0.5994–0.0927i	0.5994–0.0927i (<0.01%)(<0.01%)	0.5993–0.0927i (–0.02%)(0.0%)
	1	0.5826–0.2813i	0.5826–0.2813i (<0.01%)(<0.01%)	0.5824–0.2814i (–0.03%)(–0.04%)
	2	0.5517–0.4791i	0.5517–0.4791i (<0.01%)(<0.01%)	0.5532–0.4767i (0.27%)(0.50%)
	3	0.5120–0.6903i	0.5120–0.6905i (<0.01%)(–0.03%)	0.5157–0.6774i (0.72%)(1.9%)
	4	0.4702–0.9156i	0.4715–0.9156i (0.28%)(<0.01%)	0.4711–0.8815i (0.19%)(3.7%)
	5	0.4314–1.152i	0.4360–1.147i (1.07%)(0.43%)	0.4189–1.088i (–2.9%) (5.6%)

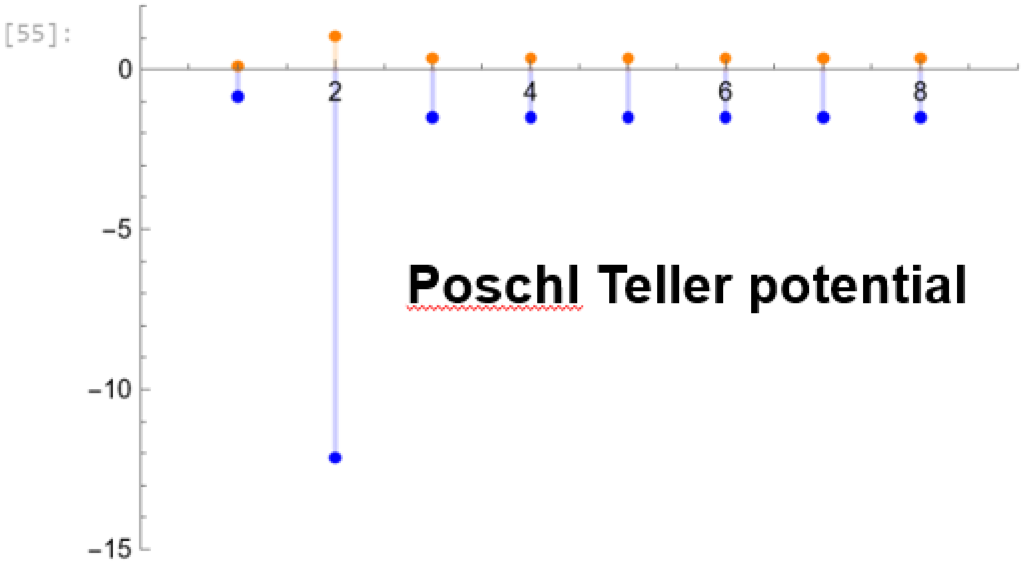
Research questions:

1. How can I tell if AIM gives me the correct value of QNF?
2. Why does AIM work for some potential and doesn't work for others?

Saad and Ismail approach

?

Asymptotic assumption: $$\frac{s_n}{\lambda_n} = \frac{s_{n-1}}{\lambda_{n-1}} = \alpha(x)$$

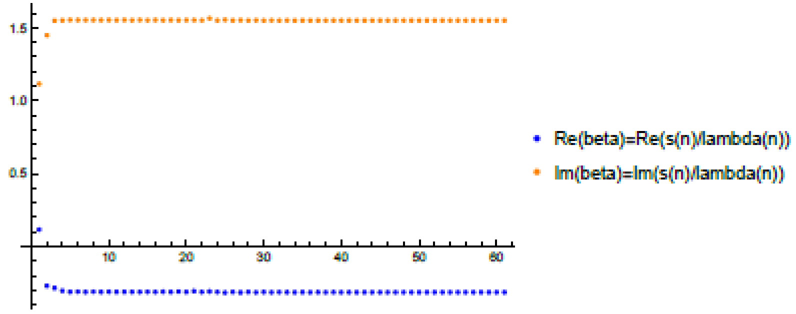


Schwarzschild BH

[59]:
$-0.180932 + 1.42328\, i$
$-0.157242 + 1.54618\, i$
$-0.199313 + 1.552\, i$
$-0.211866 + 1.55794\, i$
$-0.211866 + 1.55794\, i$
$-0.214028 + 1.55345\, i$
$-0.214077 + 1.55727\, i$
$-0.213002 + 1.55321\, i$

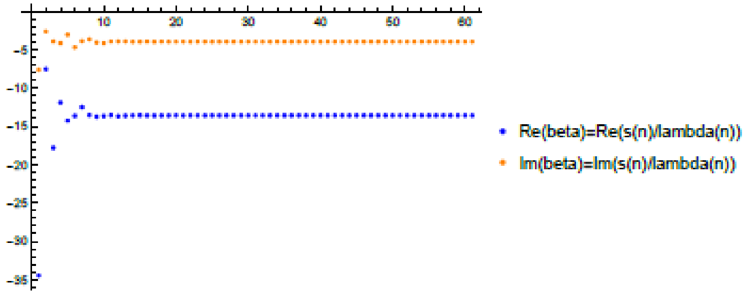
M. E. H. Ismail and N. Saad, "The asymptotic iteration method revisited," *Journal of Mathematical Physics*, vol. 61, Mar. 2020.

QNFs of Schwarzschild BH generated by AIM



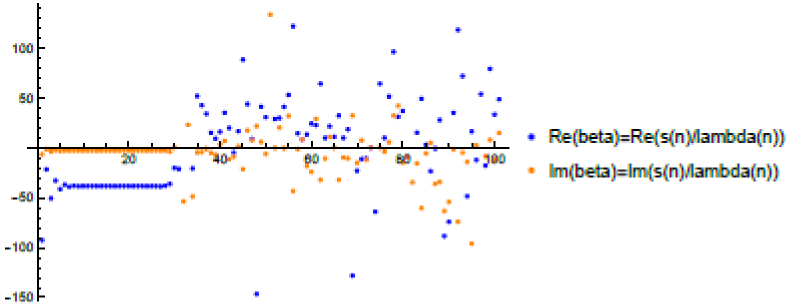
$$\omega = 0.5994 - 0.0927i,$$

s_n/λ_n for QNF of Schwarzschild spacetime with $l = 3, N = 0$ and a unit mass.



$$\omega = 0.4360 - 1.1473i$$

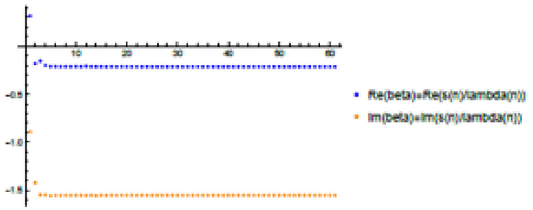
s_n/λ_n for QNF of Schwarzschild spacetime with $l = 3, N = 5$ and a unit mass.



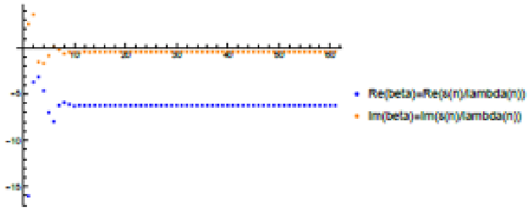
$$w = 0.2633 - 3.6654i,$$

s_n/λ_n for an inaccurate QNF generated by AIM for a Schwarzschild spacetime with $l = 3$ and a unit mass.

QNFs of Reissner-Nordström spacetime generated by AIM

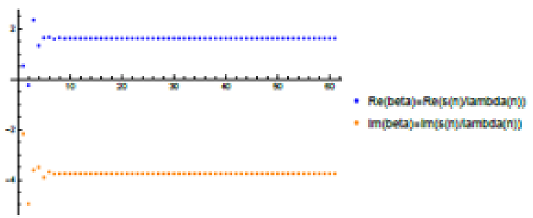


(a) N=0

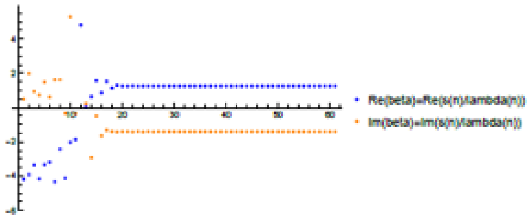


(b) N=2

s_n/λ_n using the AIM generated QNF ($\rho = -i\omega$) of the Reissner–Nordström metric for the fundamental and second overtone with $l = 2, M = 1/2$ and $i = 2$ in the Schwarzschild limit ($Q = 0$).



(a) N=0



(b) N=2

s_n/λ_n using the AIM generated QNF ($\rho = -i\omega$) of the Reissner–Nordström metric for the fundamental and second overtone with $l = 2, M = 1/2$ and $i = 2$ in the Extremal limit ($Q = 0.495$).

Batic and Nowakowski approach

Continuous Fraction Method (CFM)

$$y^{(n+2)}(x) = p_n(x)y^{(n+1)}(x) + q_n(x)y^{(n)}(x)$$

$$p_n = p_{n-1} + \frac{q'_{n-1}}{q_{n-1}}, \qquad q_n = q_{n-1} + p'_{n-1} - p_{n-1}\frac{q'_{n-1}}{q_{n-1}}$$

$$p_0 := \lambda_0, \qquad q_0 := s_0, \qquad p_{-1} = q_{-1} = 0.$$

$$\alpha(x) = \psi_0(x) = \frac{q_0(x)}{p_0(x) + \frac{q_1(x)}{p_1(x) + \frac{q_2(x)}{p_2(x)+\dots}}} := \underset{n=0}{\overset{\infty}{K}}\left(\frac{q_n(x)}{p_n(x)}\right). \qquad \alpha(x) \underset{\lim n\to\infty}{:=} \frac{s_n(x)}{\lambda_n(x)}$$

D. Batic and M. Nowakowski, "The mathematical foundations of the asymptotic iteration method," *Math. Methods Appl. Sci.*, vol. 46, no. 9, pp. 10833–10849, 2023.

Batic's Convergence theorem IV.2

(theorem IV.2): If $p_n(x) > 0$ and $q_n(x)$ does not vanish for all n then the continued fraction (5) converges (and hence AIM and CFM are valid) if and only if the series

$$\sum_{n=0}^{\infty} p_n(x) \qquad \underset{n=0}{\overset{\infty}{K}} \left(\frac{q_n(x)}{p_n(x)} \right)$$

diverges for every x.

Problems:

- Relevance
- Validity

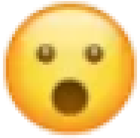
$$\alpha = \cfrac{-2j}{2x + \cfrac{2-2j}{2x + \cfrac{4-2j}{2x + \dots}}}$$

n	p_n	q_n
0	$0.09 - 2.65i$	$2.38 - 0.75i$
1	$14.55 + 7.29i$	$-19.69 + 9.60i$
2	$0.88 - 6.40i$	$6.67 + 8.31i$
3	$13.29 + 8.10i$	$-25.54 - 14.94i$
4	$3.50 - 14.72i$	$-12.25 + 54.91i$
5	$14.03 + 4.67i$	$-0.10 + 0.27i$
6	$3504.42 + 2626.59i$	$-36621.80 - 53384.40i$
7	$5.13 - 2.68i$	$183.99 - 75.54i$
8	$-9.95 - 2.28i$	$445.83 - 245.89i$

Mathematical problem

$$\mathop{K}_{n=0}^{N}\left(\frac{\gamma_n}{\gamma_n p_n}\right) = \frac{\gamma_0}{\gamma_0 p_0 + \frac{\gamma_1}{\gamma_1 p_1 + \frac{\gamma_2}{\gamma_2 p_2(x)+\dots}}} \neq \frac{1}{p_0 + \frac{1}{p_1 + \frac{1}{p_2+\dots}}} = \mathop{K}_{n=0}^{N}\left(\frac{1}{p_n}\right)$$

Counter example $q_n = -1 \text{ and } p_n = \sqrt{2}.$



$$-\frac{1}{\sqrt{2} - \frac{1}{\sqrt{2} - \frac{1}{\sqrt{2} - \dots}}}.$$

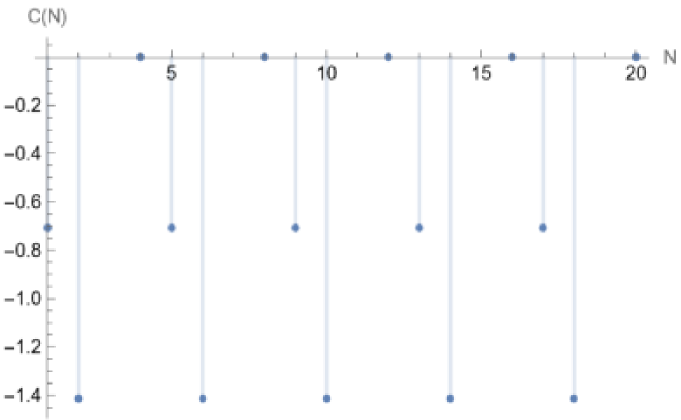
Should converge but it doesn't!

[4]:

$$-\frac{1}{\sqrt{2}}$$

$$-\frac{1}{-\frac{1}{\sqrt{2}} + \sqrt{2}}$$

$$-\frac{1}{\sqrt{2} - \frac{1}{-\frac{1}{\sqrt{2}} + \sqrt{2}}}$$



Batic's Convergence theorem IV.3

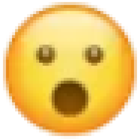
Theorem IV.3 [2] addresses the convergence of (10) from a different angle. If we consider the recurrence relation

$$\xi_n(x) - p_n(x)\xi_{n-1}(x) - q_n(x)\xi_{n-2}(x) = 0 \tag{31}$$

with $q_n(x) \neq 0$ for all $x \in I$ and for any $n \in \mathbb{N} \cup \{-1, 0\}$. If for any $x \in I$ (31) has a minimal solution $\psi_n(x)$ with $\psi_{-1}(x) \neq 0$ for all $x \in I$, then the continual fraction (10) converges pointwise to α on the defined domain.
A solution ψ_n is called a minimal solution, we recall, if

$$\lim_{n\to\infty} \frac{\psi_n}{\xi_n} = 0 \tag{32}$$

for any solution ξ_n which is not a multiple of ψ_n.



$$q(x) = \lim_{n\to\infty} \frac{4q_n(x)}{p_{n-1}(x)p_n(x)}, \qquad a_n(x) = \frac{4q_n(x)}{p_{n-1}(x)p_n(x)} - q(x),$$

$$r_\pm(x) = 1 \pm \sqrt{1 - q(x)}.$$

$q(x) < 1$ for all $x \in \mathcal{I}_1 \subseteq \mathcal{I}$ and $p_n(x)$, $q_n(x)$ are real-valued functions,

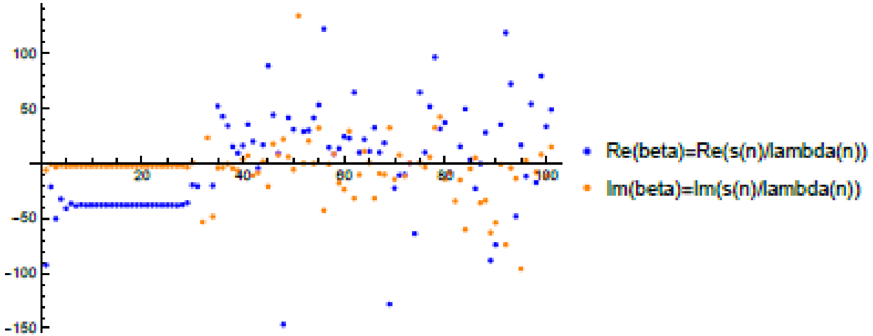
Counter example

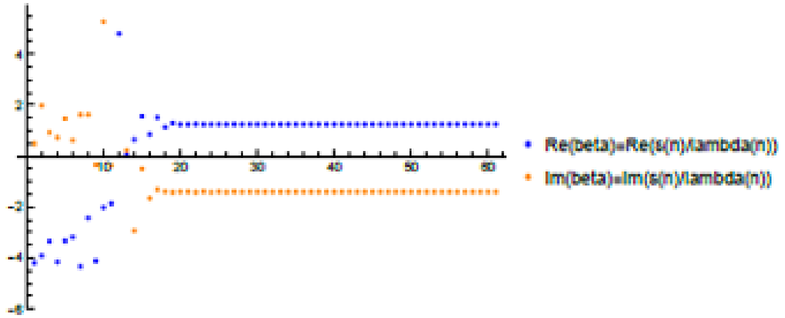
$$-\frac{1}{\sqrt{2} - \frac{1}{\sqrt{2} - \frac{1}{\sqrt{2} - \dots}}}.$$

Open question

Why some “potentials” work and some not with AIM?

$$y'' = \lambda_0(x)y' + s_0(x)y$$

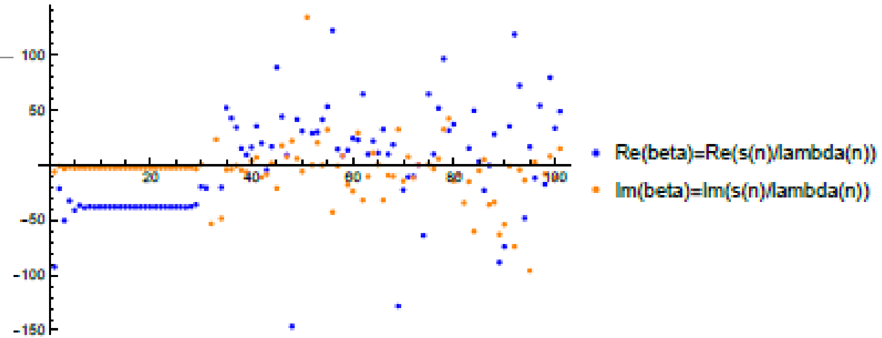


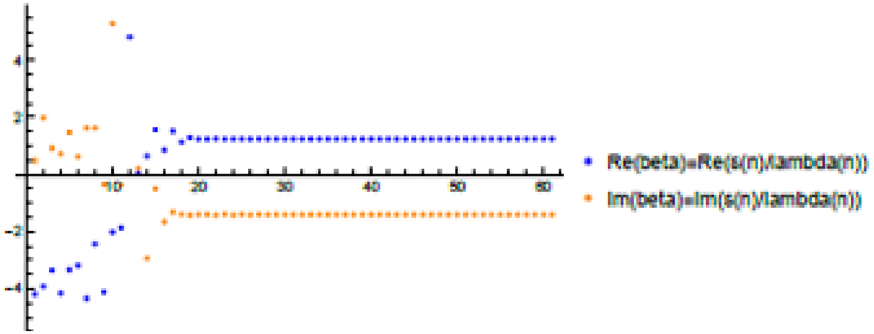


Pathological failure or just not enough iterations?

Conclusion

- SAAD ISMAIL APPROACH- EXCELLENT DIAGNOSTIC TOOL FOR CONVERGENCE BUT NOT A PROOF
- BATIC'S APPROACH- CAN THEOREM IV.3 BE GENERALIZED TO BH COMPLEX VALUED QUANTITIES? IF SO, CAN IT BE USED AS A PROOF FOR CONVERGENCE OR JUST ANOTHER DIAGNOSTIC TOOL?
- CAN CONVERGENCE BE PROVEN SOLELY BY OBSERVING λ_0 AND s_0?





$$y'' = \lambda_0(x)y' + s_0(x)y$$